Lecture 12
Basic Lyapunov theory

- stability
- positive definite functions
- global Lyapunov stability theorems
- Lasalle's theorem
- converse Lyapunov theorems
- finding Lyapunov functions

Some stability definitions

we consider nonlinear time-invariant system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$

a point $x_e \in \mathbf{R}^n$ is an *equilibrium point* of the system if $f(x_e) = 0$

x_e is an equilibrium point $\Longleftrightarrow x(t) = x_e$ is a trajectory

suppose x_e is an equilibrium point

- system is *globally asymptotically stable* (G.A.S.) if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$
  (implies x_e is the unique equilibrium point)
- system is *locally asymptotically stable* (L.A.S.) near or at x_e if there is an $R > 0$ s.t. $\|x(0) - x_e\| \leq R \Longrightarrow x(t) \to x_e$ as $t \to \infty$

- often we change coordinates so that $x_e = 0$ ($i.e.$, we use $\tilde{x} = x - x_e$)
- a linear system $\dot{x} = Ax$ is G.A.S. (with $x_e = 0$) $\Leftrightarrow \Re\lambda_i(A) < 0$, $i = 1, \ldots, n$
- a linear system $\dot{x} = Ax$ is L.A.S. (near $x_e = 0$) $\Leftrightarrow \Re\lambda_i(A) < 0$, $i = 1, \ldots, n$
  (so for linear systems, L.A.S. $\Leftrightarrow$ G.A.S.)
- there are *many* other variants on stability ($e.g.$, stability, uniform stability, exponential stability, . . .)
- when f is nonlinear, establishing any kind of stability is usually very difficult

Energy and dissipation functions

consider nonlinear system $\dot{x} = f(x)$, and function $V : \mathbf{R}^n \to \mathbf{R}$

we define $\dot{V} : \mathbf{R}^n \to \mathbf{R}$ as $\dot{V}(z) = \nabla V(z)^T f(z)$

$\dot{V}(z)$ gives $\dfrac{d}{dt}V(x(t))$ when $z = x(t)$, $\dot{x} = f(x)$

we can think of V as *generalized energy function*, and $-\dot{V}$ as the associated *generalized dissipation function*

Positive definite functions

a function $V : \mathbf{R}^n \to \mathbf{R}$ is *positive definite* (PD) if

- $V(z) \geq 0$ for all z
- $V(z) = 0$ if and only if $z = 0$
- all sublevel sets of V are bounded

last condition equivalent to $V(z) \to \infty$ as $z \to \infty$

example: $V(z) = z^T P z$, with $P = P^T$, is PD if and only if $P > 0$

Lyapunov theory

Lyapunov theory is used to make conclusions about trajectories of a system $\dot{x} = f(x)$ ($e.g.$, G.A.S.) *without finding the trajectories* ($i.e.$, solving the differential equation)

a typical Lyapunov theorem has the form:

- **if** there exists a function $V : \mathbf{R}^n \to \mathbf{R}$ that satisfies some conditions on V and $\dot{V}$
- **then**, trajectories of system satisfy some property

if such a function V exists we call it a *Lyapunov function* (that proves the property holds for the trajectories)

Lyapunov function V can be thought of as *generalized energy function* for system

A Lyapunov boundedness theorem

suppose there is a function V that satisfies

- all sublevel sets of V are bounded
- $\dot{V}(z) \leq 0$ for all z

then, all trajectories are bounded, *i.e.*, for each trajectory x there is an R such that $\|x(t)\| \leq R$ for all $t \geq 0$

in this case, V is called a Lyapunov function (for the system) that proves the trajectories are bounded to prove it, we note that for any trajectory x

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0))$$

so the whole trajectory lies in $\{z \mid V(z) \leq V(x(0))\}$, which is bounded

also shows: every sublevel set $\{z \mid V(z) \leq a\}$ is invariant

A Lyapunov global asymptotic stability theorem

suppose there is a function V such that

- V is positive definite
- $\dot{V}(z) < 0$ for all $z \neq 0$, $\dot{V}(0) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$
($i.e.$, the system is globally asymptotically stable)

intepretation:

- V is positive definite generalized energy function
- energy is always dissipated, except at 0

Proof

suppose trajectory $x(t)$ does not converge to zero.

$V(x(t))$ is decreasing and nonnegative, so it converges to, say, ϵ as $t \to \infty$.

Since $x(t)$ doesn't converge to 0, we must have $\epsilon > 0$, so for all t,
$\epsilon \leq V(x(t)) \leq V(x(0))$.

$C = \{z \mid \epsilon \leq V(z) \leq V(x(0))\}$ is closed and bounded, hence compact. So $\dot{V}$ (assumed continuous) attains its supremum on C, *i.e.*, $\sup_{z \in C} \dot{V} = -a < 0$. Since $\dot{V}(x(t)) \leq -a$ for all t, we have

$$V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t))\, dt \leq V(x(0)) - aT$$

which for $T > V(x(0))/a$ implies $V(x(0)) < 0$, a contradiction.

So every trajectory $x(t)$ converges to 0, *i.e.*, $\dot{x} = f(x)$ is G.A.S.

A Lyapunov exponential stability theorem

suppose there is a function V and constant $\alpha > 0$ such that

- V is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all z

then, there is an M such that every trajectory of $\dot{x} = f(x)$ satisfies
$\|x(t)\| \leq Me^{-\alpha t/2}\|x(0)\|$
(this is called *global exponential stability* (G.E.S.))

idea: $\dot{V} \leq -\alpha V$ gives guaranteed minimum dissipation rate, proportional to energy

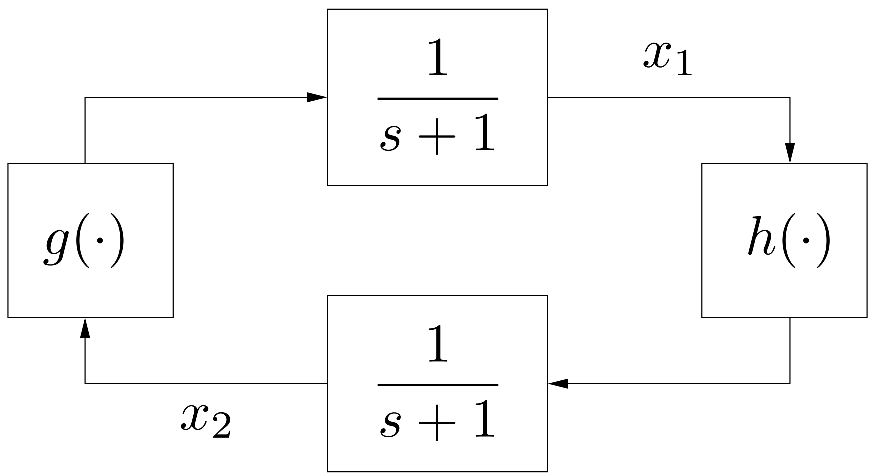
Example

consider system

$$\dot{x}_1 = -x_1 + g(x_2), \qquad \dot{x}_2 = -x_2 + h(x_1)$$

where $|g(u)| \leq |u|/2$, $|h(u)| \leq |u|/2$

two first order systems with nonlinear cross-coupling



let's use Lyapunov theorem to show it's globally asymptotically stable

we use $V = (x_1^2 + x_2^2)/2$

required properties of V are clear ($V \geq 0$, etc.)

let's bound $\dot{V}$:

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \\
&\leq -x_1^2 - x_2^2 + |x_1 x_2| \\
&\leq -(1/2)(x_1^2 + x_2^2) \\
&= -V
\end{aligned}$$

where we use $|x_1 x_2| \leq (1/2)(x_1^2 + x_2^2)$ (derived from $(|x_1| - |x_2|)^2 \geq 0$)

we conclude system is G.A.S. (in fact, G.E.S.)
without knowing the trajectories

Lasalle's theorem

Lasalle's theorem (1960) allows us to conclude G.A.S. of a system with only $\dot{V} \leq 0$, along with an observability type condition

we consider $\dot{x} = f(x)$

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- V is positive definite
- $\dot{V}(z) \leq 0$
- the only solution of $\dot{w} = f(w)$, $\dot{V}(w) = 0$ is $w(t) = 0$ for all t

then, the system $\dot{x} = f(x)$ is G.A.S.

- last condition means no nonzero trajectory can hide in the "zero dissipation" set
- unlike most other Lyapunov theorems, which extend to time-varying systems, Lasalle's theorem *requires* time-invariance

A Lyapunov instability theorem

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $\dot{V}(z) \leq 0$ for all z (or just whenever $V(z) \leq 0$)
- there is w such that $V(w) < V(0)$

then, the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ does not converge to zero (and therefore, the system is not G.A.S.)

to show it, we note that $V(x(t)) \leq V(x(0)) = V(w) < V(0)$ for all $t \geq 0$

but if $x(t) \to 0$, then $V(x(t)) \to V(0)$; so we cannot have $x(t) \to 0$

A Lyapunov divergence theorem

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $\dot{V}(z) < 0$ whenever $V(z) < 0$
- there is w such that $V(w) < 0$

then, the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ is unbounded, $i.e.$,

$$\sup_{t \geq 0} \|x(t)\| = \infty$$

(this is not quite the same as $\lim_{t \to \infty} \|x(t)\| = \infty$)

Proof of Lyapunov divergence theorem

let $\dot{x} = f(x)$, $x(0) = w$. let's first show that $V(x(t)) \leq V(w)$ for all $t \geq 0$.

if not, let T denote the smallest positive time for which $V(x(T)) = V(w)$. then over $[0, T]$, we have $V(x(t)) \leq V(w) < 0$, so $\dot{V}(x(t)) < 0$, and so

$$\int_0^T \dot{V}(x(t))\, dt < 0$$

the lefthand side is also equal to

$$\int_0^T \dot{V}(x(t))\, dt = V(x(T)) - V(x(0)) = 0$$

so we have a contradiction.

it follows that $V(x(t)) \leq V(x(0))$ for all t, and therefore $\dot{V}(x(t)) < 0$ for all t.

now suppose that $\|x(t)\| \leq R$, *i.e.*, the trajectory is bounded.

$\{z \mid V(z) \leq V(x(0)),\ \|z\| \leq R\}$ is compact, so there is a $\beta > 0$ such that $\dot{V}(z) \leq -\beta$ whenever $V(z) \leq V(x(0))$ and $\|z\| \leq R$.

we conclude $V(x(t)) \leq V(x(0)) - \beta t$ for all $t \geq 0$, so $V(x(t)) \rightarrow -\infty$, a contradiction.

Converse Lyapunov theorems

a typical *converse Lyapunov theorem* has the form

- **if** the trajectories of system satisfy some property
- **then** there exists a Lyapunov function that proves it

a sharper converse Lyapunov theorem is more specific about the form of the Lyapunov function

example: if the linear system $\dot{x} = Ax$ is G.A.S., then there is a quadratic Lyapunov function that proves it (we'll prove this later)

A converse Lyapunov G.E.S. theorem

suppose there is $\beta > 0$ and M such that each trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq Me^{-\beta t}\|x(0)\| \text{ for all } t \geq 0$$

(called *global exponential stability*, and is stronger than G.A.S.)

then, there is a Lyapunov function that proves the system is exponentially stable, $i.e.$, there is a function $V : \mathbf{R}^n \to \mathbf{R}$ and constant $\alpha > 0$ s.t.

- V is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all z

Proof of converse G.E.S. Lyapunov theorem

suppose the hypotheses hold, and define

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt$$

where $x(0) = z$, $\dot{x} = f(x)$

since $\|x(t)\| \leq Me^{-\beta t}\|z\|$, we have

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt \leq \int_0^\infty M^2 e^{-2\beta t}\|z\|^2 \, dt = \frac{M^2}{2\beta}\|z\|^2$$

(which shows integral is finite)

let's find $\dot{V}(z) = \left.\frac{d}{dt}\right|_{t=0} V(x(t))$, where $x(t)$ is trajectory with $x(0) = z$

$$\begin{aligned}
\dot{V}(z) &= \lim_{t\to 0}(1/t)\left(V(x(t)) - V(x(0))\right) \\
&= \lim_{t\to 0}(1/t)\left(\int_t^\infty \|x(\tau)\|^2\, d\tau - \int_0^\infty \|x(\tau)\|^2\, d\tau\right) \\
&= \lim_{t\to 0}(-1/t)\int_0^t \|x(\tau)\|^2\, d\tau \\
&= -\|z\|^2
\end{aligned}$$

now let's verify properties of V

$V(z) \geq 0$ and $V(z) = 0 \;\Leftrightarrow\; z = 0$ are clear

finally, we have $\dot{V}(z) = -z^T z \leq -\alpha V(z)$, with $\alpha = 2\beta/M^2$

Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to *find* a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties

one common approach:

- decide form of Lyapunov function ($e.g.$, quadratic), parametrized by some parameters (called a *Lyapunov function candidate*)
- try to find values of parameters so that the required hypotheses hold

Other sources of Lyapunov functions

- value function of a related optimal control problem
- linear-quadratic Lyapunov theory (next lecture)
- computational methods
- converse Lyapunov theorems
- graphical methods (really!)

(as you might guess, these are all somewhat related)